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LETTER TO THE EDITOR

Zeros of the partition function for the triangular lattice three-state Potts model

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Abstract. We obtain the zeros of the partition function for the isotropic triangular lattice three-state Potts model on a finite lattice. The distribution exhibits new points on the real axis which are well fitted by an algebraic equation deduced from the inversion relation and the symmetries of the anisotropic model. We identify a symmetry approximately relating all six transition points.

Recent analyses of the distribution of zeros of the partition function for the square lattice three-state Potts model clearly indicate a simple geometric locus in the ferromagnetic region (Martin 1985). In the antiferromagnetic region, however, we expect severe boundary effects which obfuscate the picture. Such boundary effects are not present in the antiferromagnetic triangular lattice model since the ground state is unique after fixing the spins at two neighbouring sites, for any lattice size.

Figure 1 contains a plot of the distribution of zeros for a 108 site triangular lattice three-state Potts model, plotted in $a=\exp(\beta)$. The partition function is given by

$$Z=\sum_{\substack{\text{configurations}\\ \{\sigma\}}} \exp\left(\beta \sum_{\text{links}} \delta(\sigma_i, \sigma_j)\right) \tag{1}$$

where σ_i takes values from $\{1, 2, 3\}$. The partition function is calculated by raising the Potts model algebraic transfer matrix to a finite power, and the distribution of zeros obtained by a Newton-Raphson analysis of the resultant high-order polynomial.

This distribution should be compared with an equivalent result for the square lattice three-state model (for example on a 10×12 lattice—see figure 2), and the exactly known locus for the triangular lattice Ising model (figure 3, from Maillard and Rammal 1983). We see that the triangular lattice three-state model indeed exhibits simple loci in both ferromagnetic and antiferromagnetic regions. The approximate intercepts of these branches with the real axis (the phase transition points) are labelled 1 and 2 respectively. We have also labelled three other intercepts and a sixth branch.

We note good numerical agreement for the three intercepts at 1, 3 and 5 with the three solutions of the exact algebraic criticality condition (Baxter *et al* 1978, Wu 1982)

$$a^3-3a-1=0. \tag{2}$$

The point 2 in the antiferromagnetic region is in good numerical agreement with Enting and Wu's (1982) candidate for an antiferromagnetic transition point.

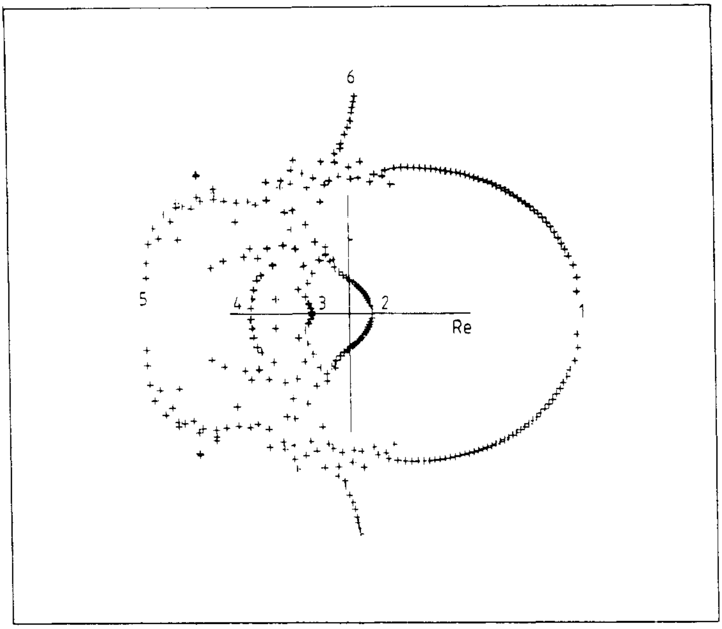

Figure 1. Zeros of the partition function in the complex $a = \exp(\beta)$ plane for a 108 site triangular lattice three-state Potts model. Distinct branches of the distribution are labelled (where appropriate at their approximate interception with the real axis) for identification in the text. In this and subsequent figures the scale is set by unit length of the positive real axis.

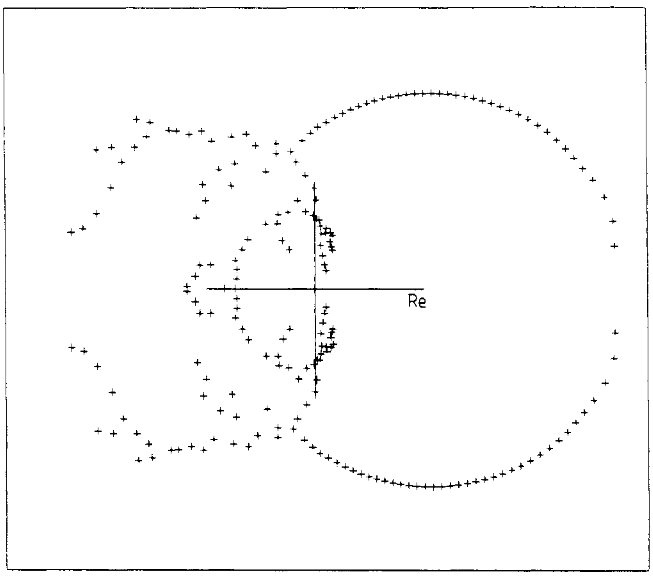

Figure 2. Zeros of the partition function in the complex $a = \exp(\beta)$ plane for a 10×12 square lattice three-state Potts model.

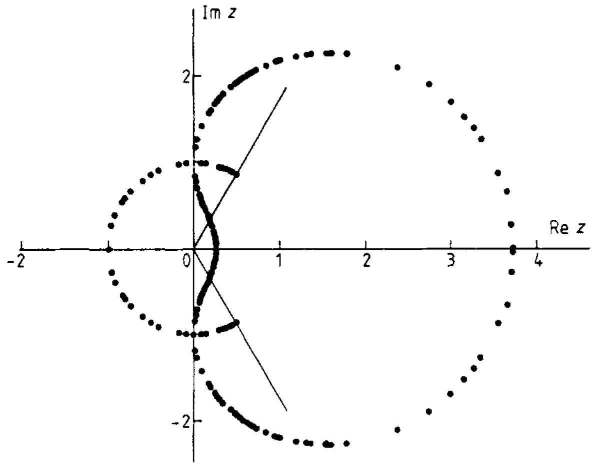

Figure 3. Limiting distribution of zeros in the (high-temperature) complex $z = (a-1)/(a+1)$ plane for the triangular lattice Ising model.

Several authors have indicated the possibility that some q-state Potts models (corresponding to the Tutte–Beraha numbers (Baxter *et al* 1976)) may be integrable like the Ising model (Baxter 1982, for instance), that is, not only at criticality, but for arbitrary temperatures. This would necessitate an algebraically based analytic structure at least on the real axis, and possibly everywhere. For example, in the triangular Ising model case the set of zeros given by figure 3 can be mapped onto the unit circle and part of the real axis by the choice of the good variable

$$k = \frac{(1+3a^2)(1-a^2)^3}{16a^6} \tag{3}$$

which is the modulus of the elliptic functions occurring in the exact solution (Syozi 1972).

In the triangular three-state case we have suggestively simple loci for the ferromagnetic and antiferromagnetic lines (1 and 2 of figure 1). We might also expect the other branches associated with the ferromagnetic line (3 and 5) to map onto 1 under an appropriate transformation and with appropriate boundary conditions (see below).

If the remaining critical points are algebraic then the automorphic structure associated with the inversion relation (Maillard and Rammal 1983) suggests, in the anisotropic triangular case, the following algebraic variety:

$$(q-2)abc + 2(ab+bc+ca) + (q-2)(a+b+c) + q^2 - 4q + 2 = 0 \tag{4}$$

(where a, b, c are the exponentiated couplings in each lattice direction) for the q-state model. In the present case this reduces to

$$a^3 + 6a^2 + 3a - 1 = 0. \tag{5}$$

Two of the three solutions are in good numerical agreement with the points 2 and 4, while the third is relatively large and could well be associated with the completion of the branch 6. We note the existence of a rational transformation of order six which maps each real point onto the others:

$$a \to (a-1)/(a+2). \tag{6}$$

We have not proved the algebraic nature of the antiferromagnetic points, but in any case a subgroup of this transformation group maps the ferromagnetic points onto each other. Furthermore, a plot of the whole distribution in the variable

$$x=(a-\omega)/(a-\omega^2)$$

where $\omega^3=1$ (so that the transformation (6) is just a rotation of $\pi/3$ in the complex plane) shows the possibility that the symmetry extends to the whole plane, at least approximately (see figure 4). The degradation of the image on rotation ($1\to2\to3\to4\to5\to6$) is as we might expect from the differing effect of boundary conditions in

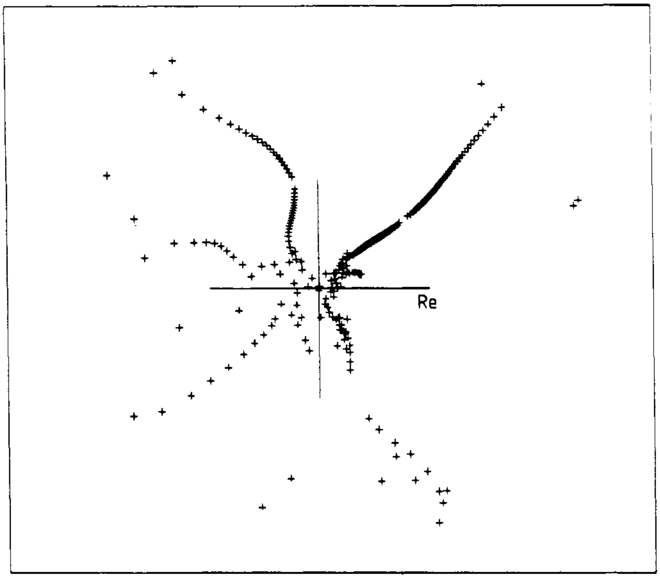

Figure 4. Zeros in the complex plane $x=(a-\omega)/(a-\omega^2)$ (where $\omega^3=1$) for the 108 site triangular lattice three-state model. The distribution is invariant under $x\to1/x^*$.

different regions of the complex plane. This would suggest that the antiferromagnetic transition is second order (Enting and Wu 1982, Wu 1982) and Monte Carlo simulations seem to indicate evidence of a first-order transition (Grest 1981).

However in the anisotropic case the conjecture (4) is difficult to support in the full range of anisotropic couplings, considering the existence of a disorder variety of the q-state triangular anisotropic model (Rujan 1984). This is because equation (4) has an intersection with the disorder variety on which the model trivialises.

In the square lattice three-state model the zero distribution (figure 2) does not suggest the existence of any simple algebraic expression which could serve as a good variable (i.e. in which the distribution is algebraically based). Contrastingly, in the triangular lattice case the existence of such a variable does seem possible. It would have to be invariant under all the symmetries of the model (inversion relation, permutation of the anisotropic couplings) and should trivialise on the disorder varieties (Baxter 1986).

We need studies on larger finite lattices to clarify the situation in the square case, since no limiting trend is yet apparent, and for the triangular model to establish if the three antiferromagnetic points are indeed algebraic.

References

Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic) p 454
—— 1986 *Free Fermion, Checkerboard and Z-invariant Lattice Models in Statistical Mechanics.* Preprint, Australian National University
Baxter R J, Kelland S B and Wu F Y 1976 *J. Phys. A: Math. Gen.* **9** 397
Baxter R J, Temperley H N V and Ashley S E 1978 *Proc. R. Soc.* A **358** 535
Enting I G and Wu F Y 1982 *J. Stat. Phys.* **28** 351
Grest G S 1981 *J. Phys. A: Math. Gen.* **14** L39
Maillard J M and Rammal R 1983 *J. Phys. A: Math. Gen.* **16** 353
Martin P P 1985 *Integrable Systems in Statistical Mechanics* ed G M d'Ariano, A Montorsi and M G Rasetti (Singapore: World Scientific) p 129
Rujan P 1984 *J. Stat. Phys.* **34** 615
Syozi I 1972 *Phase Transitions and Critical Phenomena* vol 1, ed C Domb and M S Green (London: Academic) p 322
Wu F Y 1982 *Rev. Mod. Phys.* **54** 235